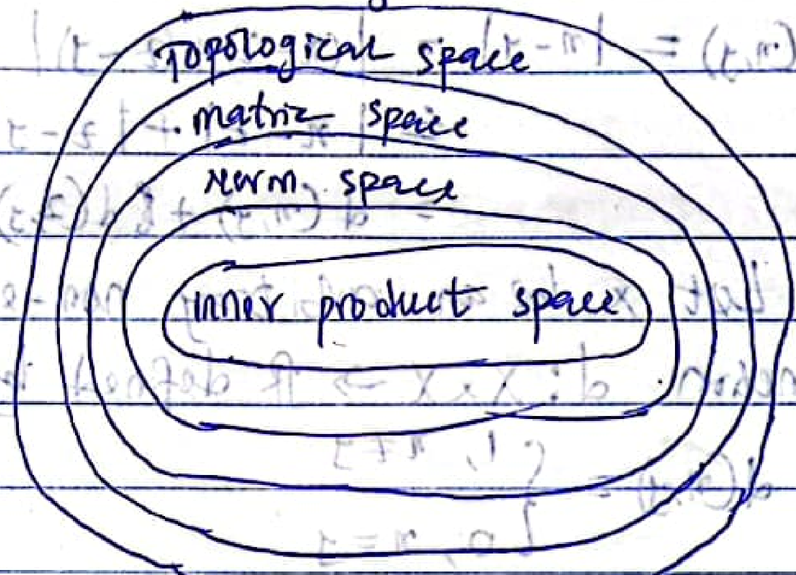
MATH 402: Functional Analysis 16/7/2023

Definition (space)

A space is a set with an added structure. In order words, a space is a structured set.

- Usually, mathematical spaces form hierarchy i.e, they form an inclusion system.



Hierarchy of maths spaces.

Definition

Let X be a non-empty set. A metric of X is a real-valued function $d: X \times X \to \mathbb{R}$ which satisfies the following conditions:

M_1 $d(x,y) \geq 0 \ \forall x,y \in X$; (non-negative)

M_2 $d(x,y) = 0 \iff x = y$; (faithfulness)

M_3 $d(x,y) = d(y,x)$; (symmetry)

M_4 $d(x,y) \leq d(x,z) + d(z,y) \ \forall x,y,z \in X$ (triangular inequality).

The pair (X,d) is called a metric space

Note that a metric is also called a distance function.

Examples 1.

The function $d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ define by $d(x,y) = |x-y|$ $\forall x,y \in \mathbb{R}$ is a metric, called the usual or standard metric.

$$d(x,y) = |x-y| = |(x-z)+(z-y)|$$
$$\leq |x-z| + |z-y|$$
$$= d(x,z) + d(z,y)$$

2. Let X be an arbitrary non-empty set. The function $d: X \times X \rightarrow \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

is a metric on X, called the trivial or discrete metric and (X,d) is called the discrete space.

3. The set l_∞ of all bounded sequence $\{x_n\}$ of real numbers with the function d defined by $d(x_n, y_n) = \sup\{x_n - y_n : n \in \mathbb{N}\}$ $\forall x_n, y_n \in l_\infty$ is a metric space.

4. The set $C[0,1]$ of all continuous real-valued function defined on $[0,1]$ with the function d given by $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$, $\forall f,g \in C[0,1]$; $x,y \in [0,1]$ is a metric space.

5. The set $C[0,1]$ with the function ~~d~~ defined by $d(f,g) = \sup\limits_{x \in [0,1]} |f(x) - g(x)|$, $\forall f, g \in C[0,1]$ is a metric space.

<u>Remark</u>: From examples 4 and 5, we can see that more than one metric can be defined on a given set.

<u>Definition</u>

Let (X,d) be a metric space. A sequence ~~of~~ (x_n) of points of X is said to converge to a point p of X if for each $\epsilon > 0$, $\exists$ a positive integer $m \ni$ $d(x_n, p) < \epsilon$, $\forall n \geq m$ ——— (1)

If (1) holds, then we ~~have~~ write

$\lim\limits_{n \to \infty} x_n = p$ or $d(x_n, p) \xrightarrow{n \to \infty} 0$ or $d(x_n, p) \to 0$ as $n \to \infty$.

The point p is called the limit point of the sequence (x_n).

<u>Definition</u>

A sequence (x_n) in a metric space (X,d) is said to be cauchy if for each $\epsilon > 0$, $\exists$ a positive α such that $d(x_n, x_m) < \epsilon$, $\forall n, m \geq \alpha$ ——— (1)

i.e $d(x_n, x_m) < \epsilon$, $\forall n, m \geq \alpha$

i.e $d(x_n, x_m) \xrightarrow{n \to \infty} 0$

Theorem:

Every convergent sequence is cauchy.

The converse of the above theorem is not always true.

Example.

Consider the space $X = [0,1]$ of the real time with the usual metric. Take $(x_n) = (1/n)_{n \in N}$ cauchy, (x_n) is a cauchy sequence of points of X and converges to 0. which is not a point of X i.e $x_n \xrightarrow{n \to \infty} 0 \notin X$.

This shows that convergence is not an inherent property of a sequence but rather that of a space.

Definition.

A metric space is said to be complete if every cauchy sequence in X converges to a point of X.

Example

Consider the space: (i) $X = (0,1]$ (ii) $Y = [0,1]$ of the real line with the usual metric. The sequence $x_n = (1/n)_{n \in N}$ is a cauchy sequence of point of X, and converges to 0. But,

i. $0 \notin X$ ii. $0 \in X$.

Definition

A subset A of a metric space (X,d) is said to be dense or everywhere dense if the closure of A is X; i.e $\bar{A} = X$.

Example

(i) The set of all rationals, $\mathbb{Q}$ is dense in the set of all real numbers, $\mathbb{R}$. i.e $\bar{\mathbb{Q}} = \mathbb{R} = (-\infty, \infty)$

(ii) Every closed interval is dense in itself.

(*) Definition

A subset A of a metric space (X,d) is said to be nowhere dense in X if the complement of the closure of A is dense in X; i.e $\overline{(\bar{A})^c} = X$.

Example

Every finite subset of (X,d) is nowhere dense on X

Definition

A set A is said to be dense in itself if every point of A is a limit point of A, i.e $A \subseteq A'$, where A' is the derived set of A; i.e the set of all limit points of A.

Definition

A set A is said to be perfect if it is closed and dense in itself.

Example

Every closed set is perfect.

Definition

An infinite set is said to be countably infinite (or denumerable or enumerable) if it is equivalent to the set N of natural numbers.

A set which is either empty, finite or countably infinite is said to be countable; otherwise it is uncountable.

Example

i. The set of all integers is countable.

ii. The set of real numbers is uncountable.

⊗ Contraction mapping principle

Definition

Let X be a nonempty set. A fixed point of a mapping $T: X \to X$ is an $x \in X$ that is mapped on to itself, that is, $Tx = x$ ——— (1)

Examples

① Let $X = \mathbb{R}$ and $T: \mathbb{R} \to \mathbb{R}$ be defined by $Tx = 1 - x$, $x \in \mathbb{R}$. Find the fixed point of T.

Soln.

By definition $x \in X$ is a fixed point of T if $x = Tx$. This implies that $x = 1 - x \Rightarrow 2x = 1 \Rightarrow x = 1/2$

② Let $X = \mathbb{R}$ and $T: X \to X$ be defined by $Tx = 1 + x$, $x \in \mathbb{R}$. Find the fixed point of T.

Soln

By $x = Tx$, we have $x = 1 + x \Rightarrow 0 = 1$ a contradiction. Hence T does not have a fixed point in X

③ Let $X = \mathbb{R}$ and define $T: \mathbb{R} \to \mathbb{R}$ by $Tx = x^2$. Find the fixed point of T.

Soln.

By $x = Tx$, $x = x^2 \Rightarrow x(1-x) = 0 \Rightarrow x = 0, 1$

Definition

Let (X, d) be a metric space. A mapping $T: X \to X$ is said to be Lipschitzian or a Lipschitz map with constant $\alpha > 0$ if for all $x, y \in X$,

$$d(Tx, Ty) \leq \alpha d(x, y) \quad \text{①}$$

If in ①, $\alpha < 1$, then the mapping T is called a contraction or Banach contraction. If $\alpha = 1$, then

$x \in [0, 1/_3] \Rightarrow 0 \le x \le 1/3$

$y \in [0, 1/3] \Rightarrow 0 \le y \le 1/3$

$\Rightarrow 0 \le x+y \le 2/3$

T is said to be non-expansive. If $\alpha = 1$ and — $d(Tx, Ty) = d(x, y)$, then T is called an isometry.

Exercise

prove that every Lipschitz mapping is uniformly continuous.

Example

If $Tx = x^2$, $x \in [0, 1/3]$, then show that T is a contraction mapping on $[0, 1/3]$ with the usual metric

soln

We are to show that $\exists \alpha \in (0,1) \ni d(Tx, Ty) \le \alpha d(x,y)$, $\forall x, y \in X = [0, 1/3]$.

now,

$$d(Tx, Ty) = |Tx - Ty| = |x^2 - y^2|$$
$$= |(x-y)(x+y)|$$
$$= |x-y||x+y|$$
$$\le 2/3 |x-y| = 2/3\, d(x,y)$$
$$\le \alpha\, d(x,y), \quad \alpha \in [2/3, 1)$$

Banach Fixed point theorem (contraction mapping theorem)

Suppose that (X, d) is a complete metric space and let $T: X \to X$ be a Banach contraction. Then T has a unique fixed point in X.

proof

Let $x_0 \in X$ be arbitrary but fixed element, and let the sequence (x_n) in X be defined iteratively as:

$x_0, x_1 = Tx_0, x_2 = Tx_1 = T^2 x_0, x_3 = Tx_2 = T^3 x_0, \ldots x_n = T^n x_0$

i.e $x_{n+1} = Tx_n, n \geq 0.$ —— (1)

Since T is a Banach contraction, then $\exists \alpha \in (0,1)$ $\ni d(Tx, Ty) \leq \alpha d(x,y) \;\forall x, y \in X$ —— (2)

By (1) and (2), we have

$$
\begin{aligned}
d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\
&\leq \alpha d(x_n, x_{n-1}) \\
&= \alpha d(Tx_{n-1}, Tx_{n-2}) \\
&\leq \alpha[\alpha d(x_{n-1}, x_{n-2})] \\
&= \alpha^2 d(x_{n-1}, x_{n-2}) \\
&\;\vdots \\
&\leq \alpha^n d(x_1, x_0)
\end{aligned}
$$

Now, by triangle inequality and the formula for the sum of a geometric series, we have for $n > m$.

$$
\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\
&\leq (\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}) d(x_0, x_1) \\
&= \alpha^m (1 + \alpha + \alpha^2 + \cdots + \alpha^{n-m-1}) d(x_0, x_1) \\
&= \frac{\alpha^m (1 - \alpha^{n-m})}{1-\alpha} d(x_0, x_1) \quad \text{—— (3)}
\end{aligned}
$$

Since $0 < \alpha < 1$, in (3), we have $1 - \alpha^{n-m} < 1$

Hence, $d(x_m, x_n) \leq \dfrac{\alpha^m}{1-\alpha} d(x_0, x_1)$ —— (4)

On the right-hand side of (4), we can make the term as small as possible by taking m as large as possible. And so, $d(x_m, x_n) \xrightarrow{n,m \to \infty} 0$.
This shows that the sequence (x_n) is Cauchy in (X,d). By the completeness of (X,d), $\exists\, u \in X$ such that $x_n \xrightarrow{n \to \infty} u$.

We now show that u is a fixed point of T.
From the triangle inequality and the fact that T is a contraction, we have

$$d(u, Tu) \leq d(u, x_n) + d(x_n, Tu)$$
$$= d(u, x_n) + d(Tx_{n-1}, Tu)$$
$$\leq d(u, x_n) + \alpha d(x_{n-1}, u) \quad \text{—— (5)}$$

Letting $n \to \infty$ in (5) gives
$d(u, Tu) \leq 0 \Rightarrow d(u, Tu) = 0$ i.e. $u = Tu$.
Suppose that v is another fixed point of T, i.e. $v = Tv$. Then,
$d(u,v) = d(Tu, Tv) \leq \alpha d(u,v)$
$\Rightarrow (1-\alpha) d(u,v) \leq 0 \Rightarrow d(u,v) = 0 \Leftrightarrow u = v$ $\square$

TOPIC Normed spaces 17/07/2023

Definition 1.1

Let X be a real or complex vector space. A norm on X is a real-valued function denoted by $\|\cdot\|$, defined for all vector $x \in X$ and satisfying the following axioms:

(N1) $\|x\| \geq 0$; and $\|x\| = 0$ if and only if $x = 0$;

(N2) $\|\alpha x\| = |\alpha| \|x\|$, where α is a scalar;

(N3) $\|x+y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$.

A normed space (or normed vector space or normed linear space) is a vector space X equipped with a norm; that is, the pair $(X, \|\cdot\|)$ is called a normed space.

It can be seen that every normed space is a metric space with respect to the metric.

$$d(x,y) = \|x-y\|; \quad x, y \in X \text{ ——— } (1)$$

The metric in (1) is called the metric induced or generated by the norm.

If in Definition 1.1, only $\|x\| \geq 0$ holds; then $\|\cdot\|$ is called a seminorm. in otherwords, a seminorm is not faithful.

Definition 1.3

A complete normed space is called a Banach space. That is, a normed space X is a Banach space if every Cauchy sequence in X converges to a point of X.

Remark 1.4

By (N3) of Definition 1.1, we have

$$\big|\,\|y\| - \|x\|\,\big| \le \|y-x\| \quad \text{---} \; (1)$$

To see (1),

Let $y = y - x + x$

$$\Rightarrow \|y\| = \|y-x+x\| \le \|y-x\| + \|x\|$$
$$= \|x-y\| + \|x\|$$

$$\Rightarrow \|y\| - \|x\| \le \|x-y\| \quad \text{---} \; (2)$$

similarly,

let $x = x - y + y \Rightarrow \|x\| = \|x-y+y\| \le \|x-y\| + \|y\|$

$$\Rightarrow \|x\| - \|y\| \le \|x-y\|$$

$$\Rightarrow \|y\| - \|x\| \ge -\|x-y\| \quad \text{---} \; (3)$$

combining (2) and (3)

$$-\|x-y\| \le \|y\| - \|x\| \le \|x-y\|.$$

$$\Leftrightarrow \big|\,\|y\| - \|x\|\,\big| \le \|x-y\| = \|y-x\|$$

Some examples of normed spaces

Example.

Let $X = \mathbb{R}^2$. For arbitrary $\bar{x} = (x_1, x_2)$ in X, define $\|\cdot\|_2 : X \longrightarrow \mathbb{R}$ by $\|\bar{x}\|_2 = (x_1^2 + x_2^2)^{1/2} = \sqrt{x_1^2 + x_2^2}$

Then, $\|\cdot\|_2$ is a norm on X.

Soln.

To see that $(\mathbb{R}^2, \|\cdot\|_2)$ is a normed space, it is enough to verify condition (N3), since condition (N1) and (N2) are obvious. To see (N3),

Let $\bar{x} = (x_1, x_2)$; $\bar{y} = (y_1, y_2)$ be arbitrary elements of X.

Recall that the schwarz inequality for finite sum is given by

$$\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}$$

Now,

$$\begin{aligned}
\|\bar{x} + \bar{y}\|_2^2 &= \|(x_1, x_2) + (y_1, y_2)\|_2^2 \\
&= \|(x_1 + y_1), (x_2 + y_2)\|_2^2 \\
&= (x_1^2 + y_1^2)^2 + (x_2^2 + y_2^2)^2 = \sum_{i=1}^{2} (x_i + y_i)^2 \\
&= \sum_{i=1}^{2} (x_i^2 + 2x_i y_i + y_i^2) \\
&\le \sum_{i=1}^{2} x_i^2 + 2\sum_{i=1}^{2} |x_i y_i| + \sum_{i=1}^{2} y_i^2
\end{aligned}$$

Hence, by ① we get

$$\|\bar{x} + \bar{y}\|_2^2 \le \sum_{i=1}^{2} x_i^2 + 2\left(\sum_{i=1}^{2} x_i^2\right)^{1/2} \left(\sum_{i=1}^{2} y_i^2\right)^{1/2} + \sum_{i=1}^{2} y_i^2$$

$$= \|\bar{x}\|_2^2 + 2\|\bar{x}\|_2 \|\bar{y}\|_2 + \|\bar{y}\|_2^2$$
$$= (\|\bar{x}\|_2 + \|\bar{y}\|_2)^2$$
$$\|\bar{x}+\bar{y}\|_2 \leq \|\bar{x}\|_2 + \|\bar{y}\|_2$$

Example

The vector space $\mathbb{R}$ of real numbers is a normed space with respect to the norm. $\|x\| = |x|$, $x \in \mathbb{R}$.

soln.

(N3) holds obviously

Example

The space l_p, $1 \leq p < \infty$, defined as
$$l_p(\mathbb{R}) = \{\bar{x} = (x_1, x_2, \ldots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^p < \infty\}$$
is a normed space with a norm (p-norm) defined by $\|\bar{x}\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$, $1 < p < \infty$.

Example.

consider the space
$$l_1 = \{x = (x_1, x_2, \ldots), x \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i| < \infty\}.$$
Which of the following is/are of l_1?

(a) $x = (1, 1/2, 1/3, \ldots)$ (b) $x_n = 1/n^2 + 1$, $n = 1, 2, 3, \ldots$

soln.

(a) First we compute the sum $\sum_{i=1}^{\infty} |x_i| = 1 + 1/2 + 1/3 + \ldots = \sum 1/n$

clearly, this series diverges, and we conclude that $x \notin l_1$.

(b) $\sum |x_i| = \sum 1/n^2+1 < \sum 1/n^2$, is a p-series with $p = 2 > 1$, and hence converges. Therefore, $x \in l_1$.

Example.

Let $X = l_2 = \{x = (x_1, x_2, \ldots) : x_i \in \mathbb{R} ; \sum |x_i|^2 < \infty\}$

Determine whether or not, the following sequences are of element of l_2: (a) $x = (1, 1/2, 1/3, \ldots)$, (b) $x_n = \frac{1}{n^2+1}$, $n = 1, 2, 3$

sol^n.

(a) $\sum |x_i|^2 = \sum 1/n^2 < \infty$

(b) $\sum |x_i|^2 = \sum \frac{1}{(n^2+2)} < \sum 1/n^4 < \infty$. Thus, $x_n \in l_2$.

⊛ Some Examples of Banach space.

Example

The Euclidean space $\mathbb{R}^n$ is a Banach Space.

Proof.

Recall that the metric on $\mathbb{R}$ is defined by

~~$d(x,y) = \|x-y\|$~~ $d(x,y) = \|x-y\|$

$$= \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.

To prove completness of $\mathbb{R}^n$, it suffices to show that every Cauchy sequence in $\mathbb{R}^n$ converges to a point of $\mathbb{R}^n$. For this, let $(x_m) = (x_1^{(m)}, x_2^{(m)}, x_3^{(m)}, \ldots,)$ be an arbitrary cauchy sequence in $\mathbb{R}^n$, then by definition of cauchy sequence, for every $\varepsilon > 0$ there exists a natural number n_0 such that $d(x_m, x_r) = \left(\sum_{i=1}^{n} (x_i^{(m)} - x_i^{(r)})^2 \right)^{1/2} < \varepsilon, (r, m > n_0)$ — ① $i = 1, 2, 3, \ldots, n$.

squaring both sides of ① gives

$$\sum_{i=1}^{n} (x_i^{(m)} - x_i^{(r)})^2 < \varepsilon^2, (m, r, > n_0)$$

$$\Rightarrow |x_i^{(m)} - x_i^{(r)}|^2 < \varepsilon^2, \quad (m, r > n_0)$$

$$\Rightarrow |x_i^{(m)} - x_i^{(r)}| < \varepsilon, \quad (m, r > n_0)$$

~~This shows that~~

This shows for each i $(1 \le i \le n)$, the sequence $(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}, \ldots)$ is a cauchy sequence of real numbers, $\mathbb{R}$. Since every cauchy sequence in $\mathbb{R}$ converges, then $x_i^{(m)} \xrightarrow{m \to \infty} x_i$, say $i = 1, 2, 3, \ldots, n$

from ①, letting $r \to \infty$, we get

$d(x_m, x) < \varepsilon$, ~~$\forall m, n >$~~ $\forall n > n_0$

This proves that $x_m \xrightarrow[n \to \infty]{} x \in \mathbb{R}^n$

Since the choice of the cauchy sequence (x_m) is arbitrary, it follows that every cauchy sequence $\mathbb{R}^n$ converges to a point of $\mathbb{R}^n$. Hence, $\mathbb{R}^n$ is a Banach space

(3) Example

The space l^∞ of all bounded sequence of complex numbers is a Banach space.

proof

Recall that a metric on l^∞ is given by
$d(x,y) = \sup_i |x_i - y_i|$; $x = (x_1, x_2, \ldots) \in l^\infty$, $y = (y_1, y_2, \ldots) \in l^\infty$

Let (x_m) be an arbitrary cauchy sequence of point of l^∞. Then, by definition of cauchy sequence, we have that given any $\varepsilon > 0$, we can find a natural number α such that for all $m, r > \alpha$,
$d(x_m, x_r) = \sup_i |x_i^{(m)} - x_i^{(r)}| < \varepsilon$, $(m, r > \alpha)$ —— (1)
$\Rightarrow |x_i^{(m)} - x_i^{(r)}| < \epsilon$, $\forall m, r > \alpha$.

Hence, for each i $(1 \le r \le n)$, the sequence $(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}, \ldots)$ is a cauchy sequence of real numbers, and hence converges to x_i (say).

That is,
$$x_i^{(m)} \xrightarrow[m \to \infty]{} x_i \text{ for each } i$$

Now, using these infinitely many limits, let $x = (x_1, x_2, x_3, \ldots)$. To see that $x \in l^\infty$, putting $r \to \infty$ in (1), we have.
$d(x_m, x) < \epsilon$, $\forall m > \alpha$

Since $(x_m) = (x_i^{(m)}) \in l^\infty$, it is bounded for each i. Therefore, there exists a real number

$\Omega > 0$ such that $|x_i^{(n)}| < \Omega \ \forall i$ max 3

Hence, by ~~definition~~ the triangular inequality, we have

$$|x_i| = |x_i - x_i^{(m)} + x_i^{(m)}|$$
$$\leq |x_i - x_i^{(m)}| + |x_i^{(m)}|$$
$$\leq \epsilon + \Omega = \Omega$$

i.e $|x_i| \leq \Omega$, putting the boundedness x_i for each i

Consequently, $x_i = x \in l^\infty$ for each i. ◻

25/07/2023

Recall that every norm space is a metric space but the converse is not always true. This means not every metric can be induced by a norm. An example is the metric on a sequence space s define by $d(x,y) = \sum_{i=1} 1/2^i \dfrac{|x_i - y_i|}{1+|x_i - y_i|}$.

For a metric to be induced by a norm, it must satisfies the property of translation invariance a given here under

Lemma (Translation invariance)

A metric d induced by a norm satisfies the following two properties:

i. $d(x+p, y+p) = d(x,y)$

ii. $d(\alpha x, \alpha y) = |\alpha| d(x,y)$

For all x, y, p in a normed space X and a scalar α.

proof

T_1: Recall that $d(x,y) = \|x-y\|$

$$d(x+p, y+p) = \|x+p-(y+p)\|$$
$$= \|x+p-y-p\|$$
$$= \|x-y\| = d(x,y)$$

T_2: $d(\alpha x, \alpha y) = \|\alpha x - \alpha y\|$
$$= \|\alpha(x-y)\| = |\alpha|\|x-y\| = |\alpha| d(x,y)$$

Definition (Subspace)

A ~~norm~~ subspace Y of a normed space X is a subspace of X considered as a vector space with the norm obtained by restricting the norm on X to Y. This norm on Y is said to be induced by the norm on X. If Y is closed in X, then Y is called a closed subspace of X.

A subspace Y of a Banach space X is a subspace of considered as a normed space.

Theorem

A subspace Y of a Banach space X is complete if and only if Y is closed in X.

Definition (Converges)

A sequence (x_n) in a normed space X is convergent if there is an element $x \in X$ such that for every $\varepsilon > 0$ $\exists$ a natural number α $\ni$

$$\|x_n - x\| < \varepsilon, \quad \forall n > \alpha \quad \text{———} \quad (1)$$

If (1) holds, then we write $\lim_{n \to \infty} \|x_n - x\| = 0$ or $x_n \to x$ as $n \to \infty$ or $x_n \xrightarrow[n \to \infty]{} x$ or $x_n \xrightarrow{n \to \infty} x$

Definition (Cauchy sequence)

A sequence (x_n) in a normed space X is said to be Cauchy if for every $\varepsilon > 0$, there exist a natural number $\gamma > 0$ $\ni$ $\|x_n - x_m\| < \varepsilon$ $\forall n, m > \gamma$.

Definition (Shauder Basis)

If a normed space X contains a sequence (e_n) with the property that for every $x \in X$, there exists a unique of scalars (α_n) such that ~~that~~ $\|x - \sum_{i=1}^{n} \alpha_i e_i\| \to 0$ as $n \to \infty$. Then (e_n) is called a shauder basis for X.

Note that e_n is the sequence whose nth term is one and the other terms are zero; that is $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, $e_3 = (0, 0, 1, 0, \ldots)$, $e_n = (0, 0, 0, \ldots, 1)$.

Finite Dimentional Normed spaces

Lemma (Linear combination)

Let $\{x_1, x_2, \ldots, x_n\}$ be a linearly independent set of vectors in a normed space X. Then there exists a number $c > 0$ such that

$$\left\| \sum_{i=1}^{n} \alpha_i \right\| \geq c \sum_{i=1}^{n} |\alpha_i|$$

(7) **Theorem** (completeness)

Every finite dimentional subspace Y of a normed space X is a Banach space. In particular every finite dimentional normed space is a Banach space.

proof

We will show that Y is complete, To this effect, we consider an arbitrary cauchy sequence (y_m) in Y and show that it is convergent

$\| \ldots y^{3}_{i=1} - \ldots x_i \ldots \| = \| r - mr \|$ ...

Let, $\dim Y = n < \infty$ and $\{e_1, e_2, \dots, e_n\}$ be a basis for Y. Then each y_m in Y has a unique representation $y_m = \alpha_1^{(m)} e_1 + \dots + \alpha_n^{(m)} e_n$. Since (y_m) is a cauchy sequence, then for every $\varepsilon > 0$, there is a natural number 7 such that

$$\|y_m - y_r\| < \varepsilon \quad \forall m, r > 7 \quad \text{——— (1)}$$

from (1) and the linear combination Lemma, there is a constant $c > 0$ such that :

$$\varepsilon > \|y_m - y_r\| = \left\| \sum_{i=1}^{n} (\alpha_i^{(m)} e_i - \alpha_i^{(r)} e_i) \right\|$$

$$\geq c \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(r)}|, \text{ when } m, r > 7 \text{ ——— (2)}$$

Dividing by c in (2), yields

$\sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(r)}| \leq 1/c \|y_m - y_r\| < \varepsilon/c$, from which

It follows that

$$|\alpha_i^{(m)} - \alpha_i^{(r)}| < \varepsilon/c; \text{ where } m, r > 7$$

This shows that each of the n sequences $(\alpha_i^{(m)}) = (\alpha_i^{(1)}, \alpha_i^{(2)}, \dots)$, $i = 1, 2, 3, \dots, n$ is cauchy in $\mathbb{R}$ or $\mathbb{C}$. Hence, it converges.

Let α_i denote the n limits. Then, using these limits, we can let $y = \sum_{i=1}^{n} \alpha_i e_i$. It is easy to see that $y \in Y$.

Moreover, $\|y_m - y\| = \left\| \sum_{i=1}^{n} \alpha_i^{(m)} e_i - \sum_{i=1}^{n} \alpha_i e_i \right\|$

$$= \left\|\sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i) e_i\right\|$$
$$\le \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i| \, \|e_i\| \quad \text{——} (3)$$

On the R.H.S of (3) $\alpha_i^{(m)} \xrightarrow{n\to\infty} \alpha_i$. Therefore, $\|y_m - y\| \xrightarrow{m\to\infty} 0$; $y_m \xrightarrow{m\to\infty} y \in Y$. This shows that (y_m) converges to a point of Y. Since (y_m) is arbitrary, It follows that Y is complete and hence a Banach space. □

Corollary

Every finite dimensional subspace Y of a normed space X is closed on X.

Definition (Equivalent norms)

A norm $\|\cdot\|_1$ on a normed space X is said to be equivalent to a norm $\|\cdot\|_2$ on X if there exists two real numbers $\alpha, \beta > 0$ such that for all $x \in X$, $\alpha\|x\|_1 \le \|x\|_2 \le \beta\|x\|_1$.

Exercise

1. Show that the norm $\|x\|$ of x is the distance from x to the origin.

Quiz: ① Show that $\phi(x) = (\sqrt{|x_1|} + \sqrt{|x_2|})^2$ does not define a norm on the vector space of all ordered pairs $x = (x_1, x_2)$ real numbers.

② Show that the discrete ~~cannot be obtained~~ metric on a vector space $X \neq \{0\}$ cannot be obtained from a norm.

③ If d is a metric on a vector space $X \neq \{0\}$ which is obtained from a norm, and $\tilde{d}$ is defined by $\tilde{d}(x,x) = 0$, $\tilde{d}(x,y) = d(x,y) + 1$, $(x \neq y)$. Show that $\tilde{d}$ cannot be obtained from a norm.

27/07/2023

Theorem

Any two norms on a finite dimensional normed space are equivalent.

proof

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be any two norms on a normed space X. Let $\dim X = n < \infty$ and $\{e_1, \ldots, e_n\}$ be a bases for X. Then, every $x \in X$ has a unique representation

$$x = \sum_{i=1}^{n} \alpha_i e_i, \text{ where } \alpha_i \text{ are scalars, } 1 \le i \le n. \quad ①$$

Then by linear combination Lemma, there exists a constant $c > 0$ such that

$$\|x\|_1 = \left\| \sum_{i=1}^{n} \alpha_i e_i \right\| \geq c \left| \sum_{i=1}^{n} \alpha_i \right| \quad ②$$

On the other hand, from ①, we have

$$\|x\|_2 = \left\|\sum_{i=1}^{n} \alpha_i e_i\right\|_2 \le \sum_{i=1}^{n} |\alpha_i| \, \|e_i\|_2$$
$$\le \max_{1\le i\le n} \|e_i\|_2 \cdot \sum_{i=1}^{n} |\alpha_i|$$
$$= \eta \sum_{i=1}^{n} |\alpha_i|, \quad \eta = \max_i \|e_i\|_2$$

i.e $\|x\|_2 \le \eta \sum_{i=1}^{n} |\alpha_i|$ —— ③

Using ②, we can write ③ as:

$\|x\|_2 \le \frac{\eta}{c} \|x\|_1$ —— ④

$\Rightarrow \frac{c}{\eta} \|x\|_2$ ~~⑤~~ $\ge \|x\|_1$ —— ⑤

By interchanging roles of $\|\cdot\|_1$ and $\|\cdot\|_2$, from ④ and ⑤, we obtain that

$\|x\|_1 \le \eta/c \, \|x\|_2 \le \eta/c \, \|x\|_1$ —— ⑥

combining ⑤ and ⑥, we get

$\alpha \|x\|_2 \le \|x\|_1 \le \beta \|x\|_1$, where $\alpha = c/\eta$ & $\beta = \eta/c > 0$

Therefore, $\|\cdot\|_2$ and $\|\cdot\|_1$ are equivalent. □

Compactness & Finite Dimension

Definition (Compactness).

A metric space X is said to be compact if every sequence in X has a convergent subsequence.

This notion of compactness is called sequential compactness of the metric space.

Lemma (C1)
Every compact subset A of a metric space X is closed and bounded.

Theorem (C2) (compactness)
In a finite dimensional normed space X, any subset A of X is compact iff it is closed and bounded.

Proof

$\Rightarrow$ Suppose that A is compact. Then, we are to show that A is closed and bounded. But, we know that every compact subset of a metric space is closed and bounded, proving the necessary direction.

$\Leftarrow$ Let A be a closed and bounded subset of X. We are to show that A is compact. Let $\dim X = n < \infty$ and $\{e_1, \ldots, e_n\}$ be a basis for X. Consider a sequence (x_m) in A. Then we are to prove that (x_m) has convergent subsequence. Since (x_m) ~~has a co~~ is a sequence of X, then every x_m in X has a representation $x_m = \alpha_1^{(m)} e_1 + \ldots + \alpha_n^{(m)} e_n$.

Since A is bounded, then (x_m) is also bounded. Hence, $\exists\, \eta > 0 \ni \|x_m\| \le \eta \ \forall m$. By linear combination lemma, we obtain

$$\eta \ge \|x_m\| = \left\| \sum_{i=1}^{n} \alpha_i^{(m)} e_i \right\| \ge c \sum_{i=1}^{n} |\alpha_i^{(m)}|, \ c > 0.$$

Therefore, the sequence of numbers $\alpha_i^{(m)}$ is bounded. Thus, by Bolzano-Weierstrass theorem, the sequence $(\alpha_i^{(m)})$ has an accumulation point, say α_i, $1 \leq i \leq n$. Following the proof of linear combination lemma, we have that (a_m) has a convergent subsequence (z_n) which converges to say $z = \alpha_i e_i$, $1 \leq i \leq n$. Since A is closed, then $z \in A$. This shows that the arbitrary sequence (a_m) has a subsequence which converges to a point of A. □

31/07/2023

Linear operator

Recall that a mapping $T: X \to X$, where X is a vector space is called a transformation. When the vector space X is made into a normed space, then $T: X \to X$ is called an operator.

2.1 Definition (Linear operator)

A linear operator T is an operator such that

(a) the domain, $D(T)$ of T is a vector space, and the range $R(T)$ lies in a vector space over the same field.

(b) for all $x, y \in D(T)$, there exist scalar α such that i. $T(x+y) = Tx + Ty$ ii. $T(\alpha x) = \alpha Tx$

By combining conditions (i), (ii) in ⑥, we can reformulate Definition 2.1 as follows

⑧ Definition 2.2.

An operator $T: D(T) \subseteq X \to Y$, where X and Y are normed spaces, is said to be linear if for all $x_1, x_2 \in D(T)$, $\exists$ scalars α, β such that

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 \quad \text{— ⑧}$$

Remark

i. If X and Y are normed spaces, and $D(T) = X$, then we write $T: X \to Y$. But, if $D(T) \neq X$ then we write $T: D(Y) \to Y$ or $T: D(T) \subset X \to Y$

ii. Every linear operator maps the zero vector to zero.

To see this condition 6(ii) of Definition 2.1 — and let $\alpha = 0$. Then

$$T(0x) = T(0) = 0 \cdot Tx = 0.$$

Alternatively, put $\alpha = \beta = 0$ in ⑧ of Defn 2.2 then $T(0x_1 + 0x_2) = 0 \cdot Tx_1 + 0 \cdot Tx_2$

i.e. $T(0) = 0$.

some examples of linear

① The identity mapping $I: X \to X$ defined by $Ix = x$, is a linear operator.

To see this, let $x_1, x_2 \in X$ and α, β be any two scalars. Since I is an identity mapping, $Ix_1 = x_1$, $Ix_2 = x_2$.

$\therefore I(\alpha x_1 + \beta x_2) = \alpha x_1 + \beta x_2 = \alpha I x_1 + \beta I x_2$, proving that linearly of I.

② The zero operator $0: X \to X$ defined by $0x = 0$ $\forall x \in X$ is linear.

proof (obvious).

③ Let X be the vector space of all polynomials on $[a,b]$. An operator $T: X \to X$ defined by $Tx(t) = x'$ is linear.

sol:

Let $x_1, x_2 \in X$, α, β be scalars.

$$
\begin{aligned}
T(\alpha x_1 + \beta x_2) = (\alpha x_1 + \beta x_2) &= \frac{d}{dx}(\alpha x_1 + \beta x_2) \\
&= \frac{d}{dx}\alpha x_1 + \frac{d}{dx}\beta x_2 \\
&= \alpha \frac{d}{dt}x_1(t) + \beta \frac{d}{dt}x_2(t) \\
&= \alpha x_1'(t) + \beta x_2'(t) \\
&= \alpha Tx_1(t) + \beta Tx_2(t)
\end{aligned}
$$

Definition (null space)

The null space of an operator T, denoted by $N(T)$ is the set of all $x \in D(T)$ such that $Tx = 0$. That is,

$$N(T) = \{x \in D(T) : Tx = 0\}.$$

Note:

The null space of T, $N(T)$ is sometimes called the kernel of T, $Ker(T)$.

③ Theorem 2.5

Let T be a linear operator. Then:

i. The range of T, $R(T)$ is a vector space

ii. If $\dim D(T) = n < \infty$, then $\dim R(T) \leq n$.

iii. The null space of, $N(T)$ is a vector space

proof

We consider $y_1, y_2 \in R(T)$ and show that $\exists$ scalars $\alpha, \beta \in K$ $\ni$ $\alpha y_1 + \beta y_2 \in R(T)$.

Now, since $y_1, y_2 \in R(T)$, then $\exists x_1, x_2 \in D(T)$ such that

$$y_1 = Tx_1 \text{ and } y_2 = Tx_2$$

and $\alpha x_1 + \beta x_2 \in D(T)$ [$\because D(T)$ is a vector space]

The linearity of T gives

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = \alpha y_1 + \beta y_2$$

$\Rightarrow \alpha y_1 + \beta y_2 \in R(T)$. This proves that $R(T)$ is a vector space.

ii. Let $y_1, y_2, \ldots, y_{n+1}$ be arbitrary elements of $R(T)$. Then, $\exists\, x_1, x_2, \ldots, x_{n+1} \in D(T)$ such that $y_1 = Tx_1$; $y_2 = Tx_2, \ldots, y_{n+1} = Tx_{n+1}$

Since $\dim D(T) = n$, this set $\{x_1, x_2, \ldots, x_{n+1}\}$ must be linearly dependent. Hence, $\exists$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ not all zeros such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{n+1} x_{n+1} = 0 \quad \text{--- (1)}$$

Since T is linear and $T0 = 0$, then applying T on both sides of (1) gives

$$T(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{n+1} x_{n+1}) = T0 = 0$$

i.e $\alpha_1 Tx_1 + \alpha_2 Tx_2 + \cdots + \alpha_{n+1} Tx_{n+1} = 0$.

This shows that $\{y_1, y_2, \ldots, y_{n+1}\}$ is linearly dependent. Since $\{y_1, \ldots, y_{n+1}\}$ was arbitrary, we conclude that $R(T)$ has no linearly independent subset of $n+1$ or more elements. By definition this means that $\dim R(T) \leq n$.

(iii.) We know that $N(T) = \{x \in D(T) : Tx = 0\}$.

Let $x_1, x_2 \in N(T) \Rightarrow T(x_1) = 0$ and $T(x_2) = 0$.

Let α, β be two scalars.

Since T is linear, then

$$T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2 = \alpha \cdot 0 + \beta \cdot 0 = 0$$

By definition, $\alpha x_1 + \beta x_2 \in N(T)$.
This proves $N(T)$ is a vector space.

Remark

From the above result, we deduce that linear operator preserves linear dependence.

Recall that a mapping $T: D(T) \to Y$ is one-one or injective iff

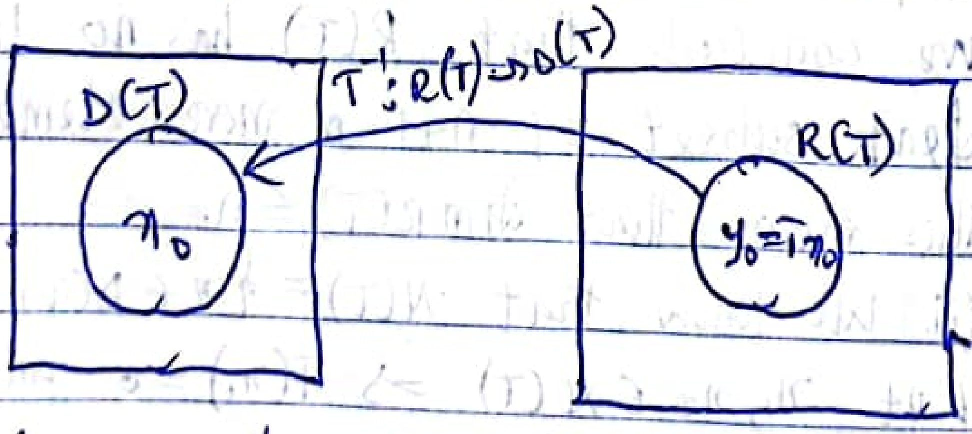
i. $Tx_1 = Tx_2 \Rightarrow x_1 = x_2$

ii. $x_1 \neq x_2 \Rightarrow Tx_1 \neq Tx_2$

from ① it follows that $\exists$ a mapping T^{-1}, called the inverse of T.

Definition

The mapping $T^{-1}: R(T) \to D(T)$ is called the inverse of T. $y_0 \longrightarrow x_0$



clearly, if T^{-1} exists, then $T^{-1}Tx = x \ \forall x \in D(T)$ and $TT^{-1}y = y \ \forall y \in R(T)$.

Theorem (Inverse operator)
Let X and Y be two vector spaces. Let $T: X \to Y$ be a linear operator, then
i. the inverse of $T^{-1}: R(T) \to D(T)$ exists if and only if $Tx = 0 \Rightarrow x = 0$.
ii. If T^{-1} exists, it is linear.

proof

i. $\Leftarrow$ suppose $Tx = 0 \Rightarrow x = 0$. Then we are to show that T^{-1} exists.
Let $Tx_1 = Tx_2$ $\forall x_1, x_2 \in D(T)$. Since T is linear, $T(x_1 + (-x_2)) = Tx_1 + T(-x_2) = Tx_1 - Tx_2 = 0$.
$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$. Hence, $Tx_1 = Tx_2$
$\Rightarrow x_1 = x_2$.
Hence, T^{-1} exists.
$\Rightarrow$ If T^{-1} exists, then we are to show that $Tx = 0 \Rightarrow x = 0$. Since T^{-1} exists, then
$Tx_1 = Tx_2 \Rightarrow x_1 = x_2$.
Take $x_2 = 0$. Then we have $Tx_1 = \cancel{T0} = 0$.
Hence, $x_1 = 0$ D.
ii. $y_1, y_2 \in R(T)$, $\exists \alpha, B \in K$.

7/8/2023

Bounded Linear Operator

Definition

Let X and Y be normed space and $T: D(T) \to Y$ a linear operator, with $D(T) \subset X$. Then T is said to be bounded if there exist a real number $c > 0$ $\ni$ $\forall x \in D(T) - \{0\}$,

$$\|Tx\| \leq c\|x\| \quad \text{--- } ①$$

To know the small c for which ① holds, observe that from ①,

$$\frac{\|Tx\|}{\|x\|} \leq c \quad \text{i.e} \quad c \geq \frac{\|Tx\|}{\|x\|}, \; x \neq 0$$

From ⊛, we see that c is atleast as long as the supremum of $\|Tx\|/\|x\|$, $x \neq 0$. so Let this quantity be denoted by $\|T\|$. That is,

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$$

putting $c = \|T\|$ in ⊛ gives

$$\frac{\|Tx\|}{\|x\|} \leq \|T\| \Rightarrow \|Tx\| \leq \|T\|\,\|x\| \quad \text{--- } ②$$

(8) Lemma

Let T be a bounded linear operator. then

i. the norm of T is given by $\|T\| = \sup_{\|x\|=1} \|Tx\|$

ii. $\|T\|$ satisfies all the four properties of norm.

proof

i. Let $\|x\| = p$ and $y = x/p$, $x \neq 0$, $p \neq 0$, then $\|y\| = \|x/p\| = \frac{\|x\|}{\|x\|} = 1$.

Using the linearity of T, we have $\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}$. Hence,

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x\neq 0} \frac{\|Tx\|}{p} = \sup_{x\neq 0} \|T(x/p)\|$$

$$= \sup_{\|y\|=1} \|Ty\| \quad \text{——— (1)}$$

so, replacing x with y in (1), gives

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

ii. (N_1) is obvious and so $\|0\| = 0$

From $\|T\| = 0$, it follows that $Tx = 0 \ \forall x \in D(T)$

$\Rightarrow T = 0$. Hence, (N_2) is verified

We also note that $\|\alpha T\| = \sup_{\|x\|=1} \|\alpha Tx\|$

$$= |\alpha| \sup_{\|x\|=1} \|Tx\|$$

$$= |\alpha| \|T\|,$$

proving (N_3): Let T_1 and T_2 be any two operators on X. Then,

$$\|T_1+T_2\| = \sup_{\|x\|=1} \|(T_1+T_2)x\|$$

$$= \sup_{\|x\|=1} \|T_1x+T_2x\|$$

$$\leq \sup_{\|x\|=1} \|T_1x\| + \sup_{\|x\|=1} \|T_2x\|$$

$$= \|T_1\| + \|T_2\|$$

i.e (N_4) holds

<u>Note</u>: the two formulae $\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}$ and $\|T\| = \sup_{\|x\|=1} \|Tx\|$ are equivalent. The two results are called the norm or operator norm of T.

<u>Some examples of Bounded Linear Operator</u>

1. The Identity linear operator $I: X \to Y$ defined by $Ix = x$ is a bounded linear operator.

<u>proof</u>

Since $Ix = x$, then $\|Ix\| \leq \|x\|$, here $c=1$.
Recall that $\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}$. Therefore,

$$\|I\| = \sup_{x\neq 0} \frac{\|Ix\|}{\|x\|} = \sup_{x\neq 0} \left(\frac{\|x\|}{\|x\|}\right) = 1$$

2. zero operator $O: X \to Y$ defined by $Ox = 0$ $\forall x \in X$.

Clearly, $\|Ox\| \le 1 \cdot \|0\|$; $c = 1$

3 Differentiation operator: Let X be the norm space of all polynomials on $J = [0,1]$ with norm given by $\|x\| = \max_{0 \le t \le 1} |x(t)|$. A differentiation operator $T: X \to X$ defined by $Tx(t) = x'(t)$ is linear but not bounded.

As an example, let $x_n(t) = t^n$, $n \in \mathbb{N}$. Then, $\|x_n(t)\| = \max_{0 \le t \le 1} |t^n| = 1$, and $Tx_n(t) = n t^{n-1}$

Hence,

$\|Tx_n(t)\| = \max_{0 \le t \le 1} |n t^{n-1}| = n$,

$\therefore \dfrac{\|Tx_n(t)\|}{\|x_n\|} = \dfrac{n}{1} = n \ne \sup_{x_n \ne 0} \dfrac{\|Tx_n(t)\|}{\|x_n(t)\|}$

Since $n \in \mathbb{N}$ is arbitrary, this shows that we cannot find a real number $c > 0$ $\ni$ $\|Tx_n\| \le c\|x_n\|$. Thus, T is not bounded

8 Theorem

If a norm space X is finite dimensional then every linear operator T on X is bounded.

proof

Let $T: X \to X$ be a linear operator, and let

$\dim X = n < \infty$ and $\{e_1, e_2, \ldots, e_n\}$ can be a b... for X. Then, any $x \in X$ has a unique representation given as :

$$x = \sum_{i=1}^{n} \alpha_i e_i \text{ for some scalars } \alpha_i \; (1 \le i \le n) \quad \text{—— ⓐ}$$

Since T is linear, then

$$\|Tx\| = \left\|\sum_{i=1}^{n} T\alpha_i e_i\right\| = \left\|\sum_{i=1}^{n} \alpha_i T e_i\right\|$$

$$\le \sum_{i=1}^{n} |\alpha_i| \, \|Te_i\|$$

$$\le \max_i \|Te_i\| \sum_{i=1}^{n} |\alpha_i| \quad \text{—— ①}$$

Recall that by linear combination lemma, if $\{x_1, \ldots, x_n\}$ is a linearly independent subset of a normed space X, then ∃ scalars $\alpha_i \; (1 \le i \le n)$ such that

$$\left\|\sum_{i=1}^{n} \alpha_i x_i\right\| \ge c \sum_{i=1}^{n} |\alpha_i|, \text{ for some } c > 0 \quad \text{—— ②}$$

setting $x_i = e_i$ for each i in ②, T gives

$$\left\|\sum_{i=1}^{n} \alpha_i e_i\right\| \ge c \sum_{i=1}^{n} |\alpha_i|$$

$$\Rightarrow \sum_{i=1}^{n} |\alpha_i| = \frac{1}{c} \left\|\sum_{i=1}^{n} \alpha_i e_i\right\|$$

Using ⓐ, it follows that

$$\sum_{i=1}^{n} |\alpha_i| \le \frac{1}{c} \|x\| \quad \text{—— ③}$$

from ① and ③, we get

$$\|Tx\| = \max_i \|Te_i\| \frac{\|x\|}{c} = 1/c \max_i \|Te_i\|\|x_i\|$$

$$= \lambda\|x\|, \text{ where } \lambda = 1/c \max_i \|Te_i\|$$

This shows that T is bounded.

Definition

Let $T: D(T) \to Y$ be an operator with $D(T) \subset X$ where X and Y are normed spaces. Then, T is said to be continuous at $x_0 \in D(T)$ if for every $\epsilon > 0$, $\exists$ a $\delta(\epsilon) = \delta > 0$ such that $|x - x_0| \|< \delta\| \Rightarrow \|T(x) - T(x_0)\| < \epsilon$ or $\|T(x) - T(x_0)\| < \epsilon$, when $|x - x_0|$

③⑧ Theorem:

Let $T: D(T) \to Y$ be a linear operator where $D(T) \subset X$ and X, Y are normed spaces. Then

i. T is continuous iff T is bounded

ii. If T is continuous at a point, then it is continuous

proof

$\Leftarrow$ suppose that T is bounded. Then we need to show that T is continuous. Note that if $T = 0$, there is nothing to show, since

for all $\epsilon > 0$, $\|Tx - Tx_0\| = \|0\| = 0 < \epsilon$
wherever $\|x - x_0\| < \delta$.
So, let $\|T\| \neq 0$ for $T \neq 0$. Then, since T is linear, for every $x \in D(T)$ $\ni$ $\|x - x_0\| < \delta$, we have
$\|Tx - Tx_0\| = \|T(x - x_0)\| \leq \|T\| \|x - x_0\| \leq \|T\|$
Taking $\delta = \epsilon / \|T\|$, we get
$$\|Tx - Tx_0\| < \|T\| \cdot \frac{\epsilon}{\|T\|} = \epsilon.$$

Since $x_0 \in D(T)$ is arbitrary, it follows that T is continuous at x_0.
$\Rightarrow$ Assume that T is continuous at $x_0 \in D(T)$
Then, for every $\epsilon > 0$, we can find a $\delta = \delta(\epsilon) > 0$
$\ni$ $\|x - x_0\| < \delta \Rightarrow \|Tx - Tx_0\| < \epsilon$, $\forall x \in D(T)$
Now, let $y \neq 0$, $\in D(T)$ and set $x = x_0 + \frac{\delta \cdot y}{\|y\|}$
then,
$x - x_0 = \delta y / \|y\|$, and $\|x - x_0\| = \|\delta y / y\| = \delta$
Since T is linear, we obtain
$$\|Tx - Tx_0\| = \|T(x - x_0)\| = \|T(\delta y / \|y\|)\|$$
$$= \delta / \|y\| \, \|Ty\| \leq \epsilon. \text{ Hence}$$
$$\|Ty\| \leq \frac{\epsilon}{\delta} \|y\| = c\|y\|$$

This proves that T is bounded.

③ Theorem

Let T be a bounded linear operator on a normed space X. Then:

i. $x_n \longrightarrow x \Rightarrow Tx_n \longrightarrow Tx$

ii. The null space $N(T)$ of T is closed.

proof

i. Let $x_n \xrightarrow{n\to\infty} x$. $\Rightarrow \|x_n - x\| \xrightarrow{n\to\infty} 0$ —— ②

Now, using the boundedness and linearity of T,

$\|Tx_n - Tx\| = \|T(x_n - x)\| \le \|T\| \|x_n - x\| \xrightarrow{n\to\infty} 0$.

by ② $\Rightarrow Tx_n \to Tx$ as $n \to \infty$.

ii. Let $x \in \overline{N(T)}$. Then there exists a sequence (x_n) in $N(T)$ such that $x_n \to x$.

$\therefore$ by (i), $Tx_n \xrightarrow{n\to\infty} Tx$. Since (x_n) is a sequence of

① $N(T)$, $Tx_n = 0$.

$\therefore Tx = 0$

$\Rightarrow x \in N(T)$, showing that $N(T)$ is closed. ▫

Definition

Two operators T_1 and T_2 are said to be equal, written as $T_1 = T_2$ if they have the same domain, i.e. $D(T_1) = D(T_2)$ and $T_1 x = T_2 x$ $\forall x \in D(T_1) = D(T_2)$

Definition (Restriction)

The restriction of an operators $T: D(T) \to Y$ to a subset $B \subset D(T)$, denoted by $T|_B$, is be operator defined by $T|_B: B \to Y$ ∋ $T|_B x = Tx$ $\forall x \in B$.

Definition (Extension)

An extension of an operator T to a subset $M \supset D(T)$ is an operator $\tilde{T}: M \to Y$ such that $\tilde{T}|_{D(T)} x = Tx \; \forall x \in D(T)$.

Theorem (Bounded linear operator).

Let $T: D(T) \to Y$ be a bounded linear operator, where $D(T)$ lies in a normed space X and Y is a Banach space. Then T has an extension $\tilde{T}: \overline{D(T)} \to Y$, where $\tilde{T}$ is a bounded linear operator of norm $\|\tilde{T}\| = \|T\|$.

proof

Let $x \in \overline{D(T)}$. Then, there exists a sequence (x_n) in $D(T)$ ∋ $x_n \xrightarrow{n \to \infty} x$; i.e $\|x_n - x\| \xrightarrow{n \to \infty} 0$. Hence, (x_n) is a Cauchy sequence.

Since T is linear & bounded, we see that $\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| = \|T(x_n - x_m)\|$

$\leq \|T\| \, \|x_n - x_m\| \longrightarrow 0$

this implies that (Tx_n) is a cauchy sequence of elements of Y. By hypothesis, Y is complete, therefore (Tx_n) converges to some point of Y, say $Tx_n \xrightarrow{n\to\infty} y$ etc.

Define $\tilde{T}$ by $\tilde{T}x = y = Tx$.

Suppose that $x_n \to x$ and $z_n \to x$. Then, $v_m \to x$, where $(v_m) = (x_1, z_1, x_2, z_2, \ldots)$.

This shows that (v_m) is a cauchy sequence, using the previous corollary. Hence, the two subsequences (Tx_n) and (Tz_n) must have the same limit.

This shows that $\tilde{T}$ is uniquely defined at all points $x \in \overline{D(T)}$. It is easy to see that $\tilde{T}$ is linear and $\tilde{T}x = Tx$ $\forall x \in D(T)$, so that $\tilde{T}$ is an extension of T. Since T is bounded and linear, we have that

$\|Tx_n\| \leq \|T\| \, \|x_n\|$.

Letting $n \to \infty$ in the above relation, we obtain $Tx_n \to Tx = y = \tilde{T}x$.

Using the continuity of $\|\cdot\|$, it follows that $\|\tilde{T}x\| \leq \|T\| \, \|x\|$.

Hence, $\tilde{T}$ is bounded and $\|\tilde{T}\| \leq \|T\|$ — ①

But, $\tilde{T}$ being an extension. This means that

$\|\tilde{T}\| \geq \|T\|$ ——— ②

from ① and ②, we obtain $\|\tilde{T}\| = \|T\|$ □.

Exercise

1. Let X and Y be normed spaces. show that a linear operator $T: X \to Y$ is bounded iff it maps bounded set in X in to bounded set in Y.

2. If $T \neq 0$ is a bounded linear operator. show that for any $x \in D(T)$ such that ~~for any $x \in D$~~ $\|x\| < 1$, we have the inequality $\|Tx\| < \|T\|$

3. Let T be a bounded linear operator from a normed space X onto a normed space Y. If there exists a positive number b such that

$$\|Tx\| \geq b\|x\|, \quad \forall x \in X,$$

Show that the inverse $T^{-1}: Y \to X$ exists and is bounded.

Definition (Functional)

Let X be a vector space. A linear functional f is a linear operator with domain in X and range in a scalar field $\mathbb{K}$. That is, the operator $f: D(f) \subset X \longrightarrow \mathbb{K}$ is called a functional.

Definition

Let X be a normed space. A linear functional $f: D(f) \to \mathbb{K}$ is said to be bounded if there exists a real number $c > 0$ such that

$$|f(x)| \leq c\|x\|, \ \forall x \in D(f) \longrightarrow (1)$$

the operator norm of f, denoted by $\|f\|$ is given by $\|f\| = \sup\limits_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup\limits_{\|x\|=1} |f(x)|$

Replacing c with the norm of f in (1), gives

$$|f(x)| \leq \|f\| \|x\|, \ \forall x \in D(f).$$

Some examples of functionals.

1. The norm $\|\cdot\|: X \to \mathbb{R}$ on a normed space X is a functional which is not linear.
2. Dot product: The dot product with one factor kept fixed defines a functional $f: \mathbb{R}^3 \to \mathbb{R}$ by $f(x) = x \cdot a = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3$, $a = \alpha_i \in \mathbb{R}^3$

The functional f is linear and bounded. In fact,

$|f(x)| = |x \cdot a| \leq \|x\| \|a\|$ —— ①

i.e $|f(x)| \leq \|x\| \|a\|$ —— ②

$\Rightarrow |f(x)| / \|x\| \leq \|a\|, \ x \neq 0$ —— ④

Taking supremum over all x of norm one in ④

$\|f\| \leq \|a\|$ —— ⑧

$\Rightarrow f$ is bounded

To determine the operator norm of f, we recall that since f is bounded, then

$|f(x)| \leq \|f\| \|x\|, \ \forall x \in \mathcal{D}(f)$

$\Rightarrow |f(x)| / \|x\| \leq \|f\|$ or $\|f\| \geq |f(x)| / \|x\|, x \neq 0$

setting $x = a$ in ⑧, yields

$\|f\| \geq |f(a)| / \|a\| = \|a\| \|a\| \Rightarrow \|f\| = \|a\|$.

For linearity, we see that for any two $x_1, x_2 \in$ and $\alpha, \beta \in \mathbb{R}$, we have

$$f(\alpha x_1 + \beta x_2) = (\alpha x_1 + \beta x_2) \cdot a$$
$$= \alpha(x_1 \cdot a) + \beta(x_2 \cdot a)$$
$$= \alpha f(x_1) + \beta f(x_2)$$

⊛ 3. Definite Integral: The function f defined by $f(x) = \int_a^b x(t)\, dt, \ \forall x \in C[a,b]$ is a bounded linear functional.

For boundedness, we see that

$|f(x)| = \left| \int_a^b x(t)\, dt \right| \leq \int_a^b 1 \cdot |x(t)|\, dt$

$\leq (b-a) \max\limits_{a \leq t \leq b} x(t) = (b-a) \|x\|$,

where $\|x\| = \max_{a \le t \le b} |x(t)|$ is a norm on $C[a,b]$.
i.e $|f(x)| \le (b-a)\|x\|$, proving that f is bounded
From which we can write $|f(x)|/\|x\| \le b-a$, $x \ne 0$.
Taking supremum over all x of norm one in the above inequality, we get

$$\|f\| = \sup_{\|x\|=1} |f(x)| \le b-a$$

i.e

$$\|f\| \le b-a \quad \text{--- (1)}$$

To obtain $\|f\| \ge b-a$, recall that since f is bounded,

$$|f(x)| \le \|f\|\|x\|, \ \forall x \in D(f) \quad \text{--- (2)}$$

$$\Rightarrow |f(x)|/\|x\| \le \|f\| \ \text{ i.e } \|f\| \ge |f(x)|/\|x\| \quad \text{--- (3)}$$

Letting $x = x_0 = 1$, we $\|x\| = \|x_0\| = 1$. Hence from (3)

$$\|f\| \ge |f(x)|/\|x\| = |f(x_0)|/\|x_0\| = |f(x_0)| = \left|\int_a^b 1\,dt\right| = b-a$$

i.e $\|f\| \ge b-a$ --- (4)

from (1) and (4), it follows that $\|f\| = b-a$

For linearity, Let $x_1, x_2 \in C[a,b]$ and $\alpha, \beta \in K$
then,

$$f(\alpha x_1 + \beta x_2) = \int_a^b (\alpha x_1(t) \cancel{dt} \cdot \beta x_2(t))\,dt$$

$$= \alpha \int_a^b x_1(t)\,dt + \beta \int_a^b x_2(t)\,dt$$

$$= \alpha f(x_1) + \beta f(x_2)$$

Definition

The set of all linear functionals on a vector space X is called the algebraic dual space. Linear functionals on X^* is called second algebraic dual space of X, denoted by X^{**}

Exercise

1- show that the set of all linear functionals on a vector space X is also a vector space

2- Find the norm of the linear functional f defined on $C[-1,1]$ by $f(x) = \int_{-1}^{0} x(t)\,dt + \int_{0}^{1} x(t)\,dt$

3. Let $f \neq 0$ be a linear functional on a vector space X and x_0 be any fixed element of $X - N(f)$, where $N(f)$ is the null space of f. Show that any $x \in X$ has a unique representation.

$x = \alpha x_0 + y$, where $y \in N(f)$.

Definition

Let X and Y be any two normed spaces. The set $B(X,Y)$ is called the vector space of all bounded linear operators from X into Y.

Exercises

Show that $B(X,Y)$ is a normed space.

Theorem

If Y is a Banach space, $B(X,Y)$ is also a Banach space.

proof

Let (T_n) be a cauchy sequence in $B(X,Y)$. We need to show that (T_n) converges to an operator in $B(X,Y)$.

since (T_n) is cauchy, then for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$\|T_n - T_m\| < \varepsilon, \ \forall n, m \geq n(\varepsilon)$. — ①

i.e $\|T_n - T_m\| \longrightarrow 0$ as $n, m \rightarrow \infty$

Using the linearity and boundedness of T_n, we have

$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\| \, \|x\| \longrightarrow 0$

as $n, m \longrightarrow \infty$

Again, from the above, we can have that

$\|T_n x - T_m x\| \leq \|T_n - T_m\| \, \|x\| < \varepsilon \|x\|$ — ②

Now, for any fixed x and given $\bar{\varepsilon} > 0$, we may choose $\varepsilon = \varepsilon_x$ so that $\varepsilon_x \|x\| < \bar{\varepsilon}$

then from ②, we obtain

$\|T_n x - T_m x\| < \bar{\varepsilon}$, proving that $(T_n x)$ is a cauchy sequence of points of Y. since Y is complete, then $\exists \ y \in Y$ such that $T_n x \rightarrow y = Tx$ (say)

The operator T is linear, since

$T(\alpha x_1 + \beta x_2) = \lim_{n \to \infty} T_n(\alpha x_1 + \beta x_2)$

$= \lim_{n\to\infty} \left(T_n(\alpha x_1 + \beta x_2)\right)$

$= \alpha \lim_{n\to\infty} T_n x_1 + \beta \lim_{n\to\infty} T_n x_2$

$= \alpha T x_1 + \beta T x_2$, showing the linearity of T.

Now, we demonstrate that T is bounded and $T_n \to T$. ~~Since~~

Since ② holds for all $m > n(\varepsilon)$, and $T_m x \xrightarrow[m\to\infty]{} Tx$, we have from the linearity of T and the continuity of the norm that $\forall n > n(\varepsilon)$ and for all $x \in X$,

$\|T_n x - Tx\| = \|T_n x - \lim_{m\to\infty} T_m x\| = \lim_{m\to\infty} \|T_n x - T_m x\|$

$< \varepsilon \|x\|$ ——— ③

This shows that $T_n - T$ is bounded. Given that T_n is bounded, then

$T = T_n - (T_n - T)$ is also bounded, since the difference of any two bounded map is bounded.

Hence, $T \in B(X, Y)$.

Moreover, taking supremum over all x of norm one, we have

$\|T_n - T\| = \sup_{\|x\|=1} \|T_n x - Tx\| < \varepsilon$

i.e. $\|T_n - T\| < \varepsilon$, $\forall n > n(\varepsilon)$.

Hence, $T_n \xrightarrow{n\to\infty} T \in B(X, Y)$

This proves that every Cauchy sequence in $B(X,Y)$ converges to a point of $B(X,Y)$. $\square$

21/08/2023 **Definition** (Dual space)

Let X be a normed space. Then the set of all bounded linear functionals on X is called the dual space of X, denoted by X'.

Theorem

The dual space X' of a normed space X is a Banach space.

proof

We know that $B(X,Y)$ is a Banach space if Y is a Banach space. Hence $X' = \{ f: X \to \mathbb{K} \mid f$ is linear and bounded$\}$ is also a Banach space. Since $B(X,Y) = B(X,\mathbb{K})$, where $Y = \mathbb{R}$ or $\mathbb{C}$ is complete. □

Definition

Let (X,d) be a metric space and $x \in X$. Then, the set $B_r(x_0) = \{ x \in X : d(x, x_0) < r, r > 0\}$ is called an open Ball in (X,d)

Definition

A subset G of (X,d) is open if for every $x_0 \in X$, we have $B_r(x_0) \subseteq G$. i.e for every point $x_0 \in X$, there is an open ball centred at x_0 which is ~~an open ball~~ contained in G.

<u>Definition</u>

Let X and Y be metric spaces. Then $T: D(T) \to Y$ with $D(T) \subset X$ is called an open mapping if for every open set in $D(T)$, the image is open in Y.

<u>Open unit ball lemma</u>:

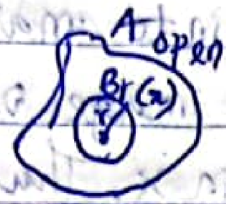
A bounded linear operator from a Banach space X onto a Banach space Y has the property that the image $T(B_1(0))$ of the open unit ball $B_1(0) \subset X$ contains an open ball about $0 \in Y$.

③ <u>Theorem</u> (Open mapping and bounded inverse theorem)

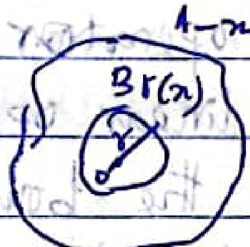
A bounded linear operator from a Banach space X on to a Banach space Y is an open mapping. Hence, if T is bijective, T^{-1} is continuous and thus bounded.

<u>proof</u>

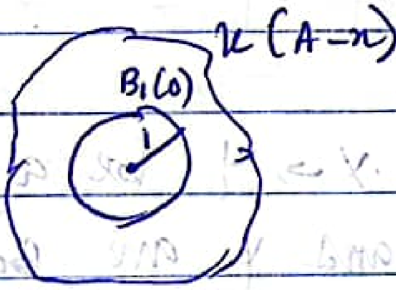
Let $A \subset X$ be open. We need to show that $T(A)$ is open in Y. For this, let $y = Tx \in Y$ for each $x \in A$. Since A is open, it contains an open ball about x. Let $r>0$ be the radius of the ball



Hence, $A-x$ contains an open ball centred at 0



Let $k = 1/r \Rightarrow r = 1/k > 0$, $k>0$. It follows that $k(A-x)$ contains an open ball with unit radius c.



Therefore, by the open unit ball Lemma, we have that

$$T(k\cdot(A-x)) = k(T(A-x)) = k(T(A)-Tx)$$

contains an open ball about $0 \in Y$. so, $T(A)-Tx$ contains an open ball centred at $0 \in Y$. Therefore, $T(A)$ contains an open ball about $Tx = y$ in Y. Since $y \in T(A)$ was arbitrary, it follows that $T(A)$ is open in Y.

Moreover, recall that a mapping T from a matrix space X into matrix space Y is continuous iff the inverse image of every open set in Y is an open set in X. Thus, if $T^{-1}: Y \to X$ exist it is continuous because T is an open mapping. By the inverse operator theorem, if T^{-1} is an inverse of a linear operator T, then it is linear. Furthermore, by the boundedness and continuity theorem, the continuity and linearity of T^{-1} imply that T^{-1} is bounded. $\triangleright$

Exercise.

Let $T: X \to Y$ be a bounded linear operator where X and Y are Banach space. If T is bijective, show that there exist positive real numbers a and b such that

$$a\|x\| \leq \|Tx\| \leq b\|x\|, \forall x \in X.$$

Definition.

Let X and Y be normed spaces and $T: D(T) \to Y$ be a linear operator with $D(T) \subset X$. Then T is called a closed linear operator if its graph: $G(T) = \{(x, y): x \in D(T), y = Tx \in Y\}$

is closed with the normed space $X \times Y$, where the two algebraic operators of vector space in $X \times Y$ are defined as usual; i.e

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and
$\alpha(x, y) = (\alpha x, \alpha y)$ $\forall x, x_1, x_2 \in X$; $y, y_1, y_2 \in Y$.
the norm on $X \times Y$ is defined as
$\|(x, y)\| = \|x\| + \|y\|$

Theorem (Closed Graph theorem)
Let X and Y be Banach spaces and $T : D(T) \to Y$ a closed linear operator where $D(T) \subset X$. then, If $D(T)$ is closed in X, the operator T is bounded.

proof

We first show that $X \times Y$ is complete. For this, Let (z_n) be a Cauchy sequence on $X \times Y$, where $z_n = (x_n, y_n)$. Then, by definition, for every $\varepsilon > 0$, there is a natural number $n(\varepsilon)$ such that

$\|z_n - z_m\| = \|(x_n, y_n) - (x_m, y_m)\|$
$= \|(x_n - x_m), (y_n - y_m)\|$
$= \|x_n - x_m\| + \|y_n - y_m\| < \varepsilon$
$\forall n, m > n_0$ — (1)

Hence, (x_n) and (y_n) are cauchy sequence in X and Y, respectively, and so, they converge, say $x_n \xrightarrow{n\to\infty} x \in X$ and $y_n \xrightarrow{n\to\infty} y \in Y$

From ①, letting $m \to \infty$, we have

$\|z_n - z\| < \epsilon, \ \forall n > N(\epsilon)$

i.e $z_n = (x_n, y_n) \xrightarrow{n\to\infty} (x, y) = z$

since the cauchy sequence (z_n) is arbitrary, then $X \times Y$ is complete.

By assumption, $G(T)$ is closed in $X \times Y$ and $D(T)$ is closed in X. Hence, $G(T)$ and $D(T)$ are complete, since every closed subset of a complete space is complete.

Now, Define a mapping $P : G(T) \to D(T)$ by $P(x, Tx) = x$. clearly, P is linear.

For boundedness, observe that

$\|P(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\| = \|x, Tx\|$.

P is bijective with the inverse mapping $P^{-1} : D(T) \to G(T)$ defined by

$P^{-1}(x) = (x, Tx)$

since $G(T)$ and $D(T)$ are complete, then by the bounded inverse theorem P^{-1} is bounded, say $\|P^{-1}(x)\| = \|x, Tx\| \leq b\|x\|$, for some $b > 0$, $\forall x \in D(T)$

Hence, T is bounded.

since

$\|Tx\| \leq \|Tx\| + \|x\| = \|(Tx, x)\| \leq B\|x\|$ □

TOPIC **Inner product & Hilbert spaces.**

Definition

Let X be a vector space. An inner product on X, denoted by $\langle \cdot, \cdot \rangle$ is a mapping of $X \times X$ in to the scalar field. that is, an inner product is a mapping $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}(\mathbb{R}$ or $\mathbb{C})$ such that for every pair of vector $x, y \in X$ there is associated a scalar written as $\langle x, y \rangle$, called the inner product of x and y such that for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{K}$, we have

(IP1) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(IP2) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

$\langle x, \beta y \rangle = \bar{\beta} \langle x, y \rangle$

(IP3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(IP4) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$

Definition

An inner product space (or pre-Hilbert space) is a vector space equipped with an inner product. That is, the pair $(X, \langle \cdot, \cdot \rangle)$ is called inner product space.

A Hilbert space is a complete inner product space.

Recall that a complete inner product space is called a Hilbert space.

Note

Every inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x, x \rangle}$ ——— (*) and a metric given by $d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y \rangle}$ ——— (**)

From (*) and (**), we conclude that every inner-product is a norm space, every Hilbert space is a Banach space, and every inner product space is a metric space. However, the converse of the above statement is not always true.

ii. In (IP3) (i.e $\langle x,y \rangle = \overline{\langle y,x \rangle}$), the bar denotes complex conjugate. Hence, if X is a real vector space, we simply write

$\langle x,y \rangle = \langle y,x \rangle$ (i.e symmetric

Exercise.

Show that a norm on an inner product space satisfies the parallelogram equality.

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

Remark.

If a norm does not satisfy the parallelogram equality, it cannot be induced/obtained from an inner product.

<u>Definition</u>

An element x of an inner product space X is said to be orthogonal to any of X if $\langle x,y\rangle = 0$.
If x is orthogonal to y, we write $x \perp y$.

Some examples of Hilbert spaces.

1. The euclidean space $\mathbb{R}^n$ is a Hilbert space with the inner product defined by ①

$$\langle x,y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n, \; x = (x_i), y = (y_i) \in \mathbb{R}^n.$$

putting $x = y$ in ①, leads to

$$\|x\| = \langle x,x\rangle^{1/2} = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2}$$

Hence, the euclidean metric is defined as

$$\|x-y\| = d(x,y) = \langle x-y, x-y\rangle^{1/2} = \left[\langle x-y, x-y\rangle\right]^{1/2}$$
$$= \left[(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2\right]^{1/2}$$

Recall that $\mathbb{R}^n$ is complete as a Banach space. Hence, It is a Hilbert space.

② The space $L^2[a,b]$, i.e the vector space of all square integrable functions defined on $[a,b]$ with with the norm

$\|x\| = \left(\int_a^b x^2(t)\, dt\right)^{1/2}$, and the inner product defined by (David Hilbert, 1912)

$\langle x,y\rangle = \int_a^b x(t)\, y(t)\, dt$, is a Hilbert space.

③ Hilbert sequence space l^2 (also called the little l) is a Hilbert space with the inner-product defined by

$\langle x,y\rangle = \sum_{i=1}^{\infty} x_i y_i$

The norm l^2 is given by $d(x,y) = \langle x-y, x-y\rangle^{1/2}$

~~Hilbert spaces~~ $\|x\| = \langle x,x\rangle^{1/2} = \left(\sum_{i=1}^{\infty} x_i^2\right)^{1/2}$

The completeness of l^2 follows from the completeness of l^p with $p=2$.

④ The space l^p with $p \neq 2$ is not a Hilbert space. This means that the norm on l^p cannot be induced by an inner product, which means that the norm on l^p does not satisfy the parallelogram equality:

example

To see that the norm on l^p fails to satisfy the parallelogram equality, let $x = (1,1,0,0,\ldots) \in l^p$, $y = (1,-1,0,0,0,0,\ldots) \in l^p$. Then we have

$\|x\| = \langle x,x\rangle^{1/p} = (x_1^2 + x_2^2 + \ldots)^{1/p} = (1^2+1^2+0^2+0^2+\ldots)^{1/p}$
$= 2^{1/p}$

$\|y\| = \langle y,y\rangle^{1/p} = (y_1^2 + y_2^2 + \ldots)^{1/p} = 2^{1/p}$ — ②

$\|x+y\| = \langle x+y, x+y\rangle^{1/p} = (\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle)^{1/p} = 2^{1/p} = \|x-y\|$ — ③

Hence, the parallelogram law: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ becomes.

$2^2 + 2^2 = 2(2^{1/p} * 2^{1/p}) = 2(2 . 2^{2/p})$

i.e $8 = 4 . 2^{2/p}$

Hence, if $p \neq 2$, the parallelogram equality fails. This means that $\|.\|_p$ cannot be generated by an inner product.

⊛ ⑤ The space $C[a,b]$ of all continuous functions on $[a,b]$ is not an inner product space, hence not a Hilbert space.

Lemma (Orthogonality lemma).
Let M and Y be any two subspaces of an inner product space X, with M complete and $x \in X$ fixed. Then $z = x - y$ is orthogonality to Y.

Definition
A vector space X is said to be the direct sum of its subspaces A and B, written as $x = A \oplus B$, if every $x \in X$ has a unique representation $x = a + b$, where $a \in A$ and $b \in B$.

Theorem (Projection theorem / Direct sum)
Let Y be any closed subspace of a Hilbert space H. Then, $H = Y \oplus Y^{\perp}$

proof

Since H is complete and Y is closed in H, then Y is also complete. Therefore, by orthogonality lemma, there exists $y \in Y \ni z = x - y$ or $x = z + y$, where $z \in Y^{\perp}$ —— ①

To see uniqueness of ①, suppose that $x = z + y = z_1 + y_1$, where $y, y_1 \in Y$ and $z, z_1 \in Y^{\perp}$

then $y - y_1 = z_1 - z$

Since $y - y_1 \in Y$ and $z_1 - z \in Y^{\perp}$, it follows that $y - y_1 \in Y \cap Y^{\perp} = \{0\}$, $\Rightarrow y - y_1 \in \{0\}$

$\Rightarrow y - y_1 = 0 \quad \therefore y = y_1$

Similarly, $z_1 - z \in Y \cap Y^{\perp} \Rightarrow z_1 = z$

This proves that the representation ① is unique

Note: that: the vector y in ① is called the orthogonal projection of x on Y. eq ① defines a mapping $P: H \to Y$, $x \longmapsto y$

i.e $P(x) = y \in Y$. The mapping P is called the projection operator of H onto Y.

Note: If $P: H \to Y$ is a projection operator, then

i. P is a bounded linear operator

ii. P maps H onto Y; i.e $P(H) = Y$

iii. P maps Y onto itself

iv. P is idempotent; i.e $P^2 = P$. Hence $\forall x \in H$

Definition

A linear operator $P: H \to H$ is called a projection operator if there exists a closed subspace Y of H such that Y is the range of P, $Y^{\perp}$ is the null space of P and $P|_Y$ is the Identity operator.

Definition

Let $T: H_1 \to H_2$ be a bounded linear operator, H_1 and H_2 are Hilbert space. Then the adjoint operator T^* of T is the operator

$T^*: H_2 \to H_1$ such that $\langle Tx, y \rangle = \langle y, T^* x \rangle$

⊛ **Definition**

A bounded linear operator $T: H \to H$ is said to be:

i. self-adjoint or Hermitian if $T^* = T$

ii. unitary if T is bijective and $T^* = T^{-1}$

iii. normal if $T^* T = T T^*$

Ⓐ It is easy to see that if T is self-adjoint, then it is normal Ⓑ Similarly, every unitary operator is normal.

proof

i. Suppose T is self-adjoint that is,

$$T^* = T \text{ ——— } ①, \quad T^* T = T \text{ ——— } ②$$

$$T T^* = T^2 \text{ ——— } ③, \quad T^* T = T T^*$$

ii. Suppose $T^* = T^{-1}$, then we want to show that $T^* T = T T^*$

$$T^* = T^{-1}$$

$$T^* T = T^{-1} T \Rightarrow T^* T = I \text{ ——— } ①$$

Similarly,

$$T T^* = T T^{-1} = I \text{ ——— } ②$$

Hence,

$$T^* T = T T^*$$

Remark

Recall that the Hilbert adjoint operator T^* of T is defined by

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{---} \; \circledast$$

If T is self-adjoint, then $\circledast$ becomes

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

Theorem

A bounded linear operator $P: H \to H$ on a Hilbert space H is a projection if and only if P is self-adjoint and indempotent.

proof

Suppose that P is a projection on H and let $P(H) = Y$, where Y is a closed subspace of H. Then $P^2 = P$ becomes for all all $x \in H$, and $Px = y$, we get

$$P^2x = P(Px) = Py = Px \quad \text{---} \; (1)$$

i.e P is idempotent

moreover, let $x_1 = y_1 + z_1$, $x_2 = y_2 + z_2$ ——— (2)

where $y_1, y_2 \in Y$ and $z_1, z_2 \in Y^\perp$. Then,

$$\langle y_1, z_2\rangle = \langle y_2, z_1\rangle \quad \text{---} \; (3)$$

Now consider:

$$\langle Px_1, x_2\rangle = \langle y_1, y_2 + z_2\rangle.$$

$$= \langle y_1, y_2 \rangle + \langle y_1, z_2 \rangle$$
$$= \langle y_1, y_2 \rangle, \text{ using } ③$$
$$= \langle y_1 + z_1, y_2 \rangle$$
$$= \langle x_1, y_2 \rangle, \text{ using } ②$$

Conversely, suppose that P is idempotent and self-adjoint. That is,

$$P^2 = P = P^* . \text{ ——— } ④$$

Let $Y = P(H)$. Then, for all $x \in H$

$x = Px + (I - P)x$, where I is the identity operator on H.

We now, consider for any $v \in H$

$$\langle Px, (I-P)v \rangle = \langle x, P(I-P)v \rangle, \text{ using } ④$$
$$= \langle x, Pv - P^2 v \rangle$$
$$= \langle x, Pv - Pv \rangle$$
$$= \langle x, 0 \rangle = 0$$

This demonstrates that $Y = P(H) \perp (I-P)(H)$.

We observe further that

$(I - P)Px = Px - P^2 x = Px - Px = 0, \forall x \in H.$

Since $Px \in Y = P(H)$, it follows that

$$Y \subset N(I - P) \text{ ——— } ⑤$$

Similarly, $(I-P)x = 0 \Rightarrow Px = x, \forall x \in H.$

This implies that $N(I - P) \subset Y$ ——— ⑥

Therefore, from ⑤ and ⑥ we have that

$Y = N(I-P)$. Hence $Y^{\perp} = N(P)$ — ⑦

Since $Y^{\perp}$ is closed, it implies that $N(I-P)$ is closed.

Now, we need to show that $P|_Y$ is the identity operator. For this, let $y = Px$. Then,

$Py = P^2x = P(Px) = Px = y$.

Therefore, $P|_Y$ is the identity operator on H. — consequently, P is a projection operator □.

Lemma (Schwarz inequality)

An inner product and the corresponding norm satisfy the inequality

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|$$

Theorem

For ~~every~~ any projection P on a Hilbert space H,

i. $\langle Px, x\rangle = \|x\|^2$

ii $P \geq 0$

iii. $\|P\| \leq 1$, $\|P\| = 1$ if $P(H) \neq \{0\}$

proof

i. Let P be a projection on H. Then,

$$\langle Px, x\rangle = \langle P^2x, Px\rangle = \langle Px, Px\rangle = \|Px\|^2 \geq 0 \text{ — ①}$$

from ①, using Schwarz inequality, we get

$$\|Px\|^2 = \langle Px, Px\rangle = \langle Px, x\rangle$$
$$\leq \|Px\|\,\|x\|$$
$$\Rightarrow \frac{\|Px\|^2}{\|Px\|} \leq \|x\| \Rightarrow \|Px\| \leq \|x\|$$
$$\Rightarrow \frac{\|Px\|}{\|x\|} \leq 1 \quad \text{——— (8)}$$

Taking suprimum over all x of norm one in (8), we get $\|P\| = \sup_{\|x\|=1} \|Px\| \leq 1$ i-e $\|P\| \leq 1$

Also,

$\dfrac{\|Px\|}{\|x\|} = \dfrac{\|x\|}{\|x\|} = 1$ from which we obtain that

$\|P\| = 1$. □

Representation of functionals on Hilbert spaces

Riesz theorem

Every bounded linear functional f on a Hilbert H can be represented in terms of an inner product given by

$$f(x) = \langle x, z\rangle \quad \text{——— (2)}$$

where z is uniquely determined on by f and has the norm $\|z\| = \|f\|$.

Recall

(*) **Definition**

An element x of a Hilbert space H is said to be orthogonal to a subspace Y of H if $\langle x,y\rangle = 0 \ \forall y \in Y$. i.e $x \perp y, \forall y \in H$

The set of all vectors in H that are orthogonal to a subspace Y of H is denoted by Y; that is

$$Y^{\perp} = \{x \in H : \langle x,y\rangle = 0 \ \forall y \in Y\}.$$

(*) **Definition**

A subset M of an inner product space X is said to be convex if for all $x,y \in X$, there exist a real number α such that $\alpha x + (1-\alpha)y \in N$ — (*)

In order words, a set N is convex if for any two points $x,y \in M$, the line segment joining x and y lies entirely in M.

E.g

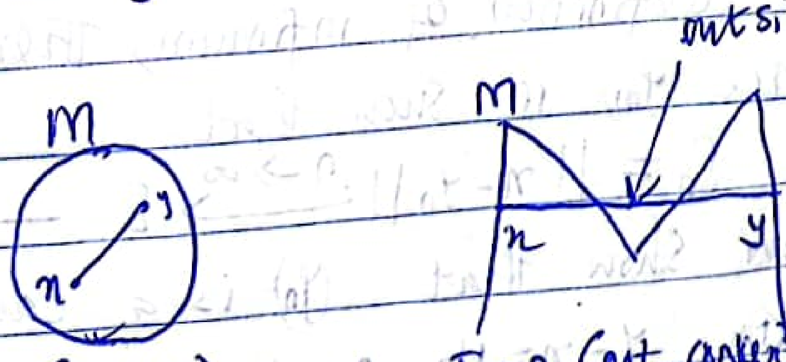


Fig. 1 (convex) Fig 2 (not convex)

Note from (*) that the coefficient of x & y sum up to one i.e $\alpha + (1-\alpha) = 1$.

Recall that a subset Y of a vector space X is a subspace of X if it is a vector space under the operation on X. That is, for all $x, y \in Y$, $\exists$ scalars α, β s.t. $\alpha x + \beta y \in Y$.

Remark

Every subspace Y of an inner product space is a convex set.

Theorem (Minimization theorem)

Let X be an inner product space and $M \neq \phi$ be a convex subset of X which is complete. Then, for every given $x \in X$, there exists a unique $y \in M$ such that

$$\delta = \inf_{\tilde{y} \in M} ||x - \tilde{y}|| = ||x - y||$$

proof

Existence

By the definition of infimum, there exist a sequence (y_n) in M such that

$$\delta_n = ||x - y_n|| \xrightarrow{n \to \infty} \delta \quad \text{——— (1)}$$

We now show that (y_n) is a Cauchy sequence. so, letting $y_n - x = v_n$, $n \in \mathbb{N}$, we have $||y_n - x|| = ||v_n|| = \delta_n$ ——— (*)

and

$\|v_n + v_m\| = \|y_n - x + y_m - x\| = \|y_n + y_m - 2x\| = 2\|\tfrac{1}{2}$

$= \tfrac{1}{2}\|\tfrac{1}{2}(y_n + y_m) - x\|$.

$\geq 2\delta$, ——— ②

because $\tfrac{1}{2}(y_n + y_m) \in M$.

Moreover, we have that

$y_n - y_m = (v_n + x) - (v_m + x) = v_n - v_m$

Recall that the ||grm equality is given by

$\|y_n - y_m\|^2 + \|y_n + y_m\|^2 = 2(\|y_n\|^2 + \|y_m\|^2)$

Therefore, by the ||grm equality, we have

$\|y_n - y_m\|^2 = \|v_n - v_m\|^2 = -\|v_n + v_m\|^2 + 2(\|v_n\|^2 + \|v_m\|^2)$

$\leq -(2\delta)^2 + 2(\delta_n^2 + \delta_m^2)$

$= -4\delta^2 + 2(\delta_n^2 + \delta_m^2) \xrightarrow{n,m\to\infty} -4\delta^2 + 4\delta^2 = 0$ — ③

It follows that the sequence (y_n) is cauchy in M.

Since M is complete by assumption, $\exists\, y \in M$ s.t.

$y_n \xrightarrow{n\to\infty} y$.

Since $y \in M$, we have that $\|x - y\| \geq \delta$. Also,

by ①,

$\|x - y\| = \|x - y_n + y_n - y\| \leq \|x - y_n\| + \|y_n - y\|$

$= \delta_n + \|y_n - y\| \xrightarrow{n\to\infty} \delta$

This shows that $\|x - y\| = \delta$, for some $y \in M$.

For uniqueness

Assume that $\exists\, y_0 \in M$ such that

$\|x - y\| = \delta$. and $\|x - y_0\| = \delta$. We need to demonstrate that $y = y_0$.

again, by the ||grm equality, we get

$$\|y-y_0\|^2 = \|(y-x) - (y_0-x)\|^2$$
$$= 2\|y-x\|^2 + 2\|y_0-x\|^2 - \|(y-x)+(y_0-x)\|^2$$
$$= 2\delta^2 + 2\delta^2 - 2^2\|\tfrac{1}{2}(y+y_0) - x\|^2 \quad \text{--- (4)}$$

Note that on the RHS of (4), $\tfrac{1}{2}(y+y_0) \in M$, so that $\|\tfrac{1}{2}(y+y_0) - x\| \geq \delta$. Hence

$\|y-y_0\|^2 = 2\delta^2 + 2\delta^2 - 4\delta^2 = 0$, proving that $y = y_0$.

4/9/2023 (8) Riesz theorem

Every Bounded linear functional f on Hilbert space H can be represented in terms of an inner product given by

$$f(x) = \langle x, z \rangle \quad \text{--- (*)}$$

Where z is uniquely determined by f and has the norm $\|f\| = \|z\|$ --- (**)

Proof

If $f = 0$, then (*) and (**) hold directly if $z = 0$.

Suppose $f \neq 0$. Then $N(f)^\perp \neq \{0\}$. Hence, $N(f)^\perp$ ~~then~~ contains a vector $z_0 \neq 0$ and $z_0 \perp N(f)$.

consider the auxiliary function v defined by

$$v = f(x) z_0 - f(z_0) x \quad \text{--- (1)}$$

Where $x \in H$ is arbitrary. Applying f on both of (1) gives

$f(y) = f(x) f(z_0) - f(z_0) f(x) = 0$

$\Rightarrow y \in N(f)$. Since $z_0 \perp N(f)$, this yields

$$0 = \langle y, z_0 \rangle = \langle f(x) z_0 - f(z_0) x, z_0 \rangle$$
$$= \langle f(x) z_0, z_0 \rangle - \langle f(z_0) x, z_0 \rangle$$
$$= f(x) \langle z_0, z_0 \rangle - f(z_0) \langle x, z_0 \rangle$$

since $z_0 \neq 0$, then $\langle z_0, z_0 \rangle = \|z_0\|^2 > 0$. Hence, solving for $f(x)$ from the above equation, produces

$$f(x) = \frac{f(z_0) \langle x, z_0 \rangle}{\langle z_0, z_0 \rangle} = \left\langle x, \frac{\overline{f(z_0)}\, z_0}{\langle z_0, z_0 \rangle} \right\rangle$$
$$= \langle x, z \rangle, \quad z = \frac{\overline{f(z_0)}}{\langle z_0, z_0 \rangle} z_0.$$

Hence, ⊛ is proved.

We now prove that z in ⊛ is unique. For this assume that $f(x) = \langle x, z_1 \rangle = \langle x, z_2 \rangle$.

Then $\langle x, z_1 \rangle - \langle x, z_2 \rangle = 0, \ \forall x \in H$.

$\Rightarrow \langle x, z_1 - z_2 \rangle = 0, \ \forall x \in H.$ — ⓐ

in particular, setting $x = z_1 - z_2 \in H$, we have $\langle z_1 - z_2, z_1 - z_2 \rangle = \|z_1 - z_2\|^2 = 0 \Rightarrow z_1 = z_2$

Hence, the choice of z in ⊛ is unique.

Moreover, we show that $\|z\| = \|f\|$

putting $x = z$ in ⊛ (i.e from $f(x) = \langle x, z \rangle$, it follows ~~that~~

$$f(z) = \langle z, z \rangle = \|z\|^2 \leq \|f\| \, \|z\|.$$

Since $z_0 \neq 0$, then dividing both sides of the above equation by $\|z\|$, gives

$$\|z\| \leq \|f\| \longrightarrow (3)$$

To show that $\|z\| \geq \|f\|$, we apply Cauchy-Schwarz inequality to (**) as

$$|f(x)| = |\langle x, z\rangle| \leq \|x\| \|z\|$$

$$\Rightarrow \frac{|f(x)|}{\|x\|} \leq \|z\|.$$

Taking supremum over all x of norm one, yields

$$\sup_{\|x\|=1} |f(x)| \leq \sup_{\|x\|=1} \|z\|$$

$$\Rightarrow \|f\| \leq \|z\| \longrightarrow (4)$$

From (3) & (4), it follows that $\|z\| = \|f\|$ □.

Extension Results for linear functionals:

Definition

Let X be a vector space. A linear functional is a real-valued functional $p: X \rightarrow \mathbb{R}$ which is sub additive, i.e for all $x, y \in X$,

$$p(x+y) \leq p(x) + p(y)$$

and positive homogeneous, that for all $x \in X$, $\exists$ a scalar α such that $p(\alpha x) = \alpha p(x)$.

(*) Hahn - Banach theorem (Generalized).

Let X be a vector space over $\mathbb{K}$ and p a sublinear functional on X. Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies

$$|f(x)| \leq p(x), \ \forall x \in X.$$

Then f has a linear extension $\tilde{f}$ from Z to X satisfying $|\tilde{f}(x)| \leq p(x), \ \forall x \in X.$

(*) Hahn - Banach theorem (Normed space)

Let f be a bounded linear functional on a subspace Z of a normed space X. Then there exists a bounded linear function $\tilde{f}$ on X which is an extension of f from Z to X and enjoy the same norm, i.e

$$\|f\|_Z = \|\tilde{f}\|_X \text{ where}$$

$$\|f\|_Z = \sup_{\|x\|=1} |f(x)| \text{ and } \|\tilde{f}\|_X = \sup_{\|x\|=1} |\tilde{f}(x)|.$$

proof

If $Z = \{0\}$, then $f=0$ and the extension is obviously $\tilde{f} = 0$. So, let $Z \neq \{0\}$. Since $f: Z \to \mathbb{K}$ is bounded, then

$$|f(x)| \leq \|f\|_Z \|x\|, \ \forall x \in Z \text{ ——— } (1)$$

Let $|f(x)| \leq p(x), \ \forall x \in Z$ ——— (2)

From 1 and 2, we notice that p is of the form

$$p(x) = \|f\|_Z \|x\|, \quad \forall x \in Z \quad \text{——— (3)}$$

To apply Hahn - Banach theorem for vector spaces, we need to prove that the mapping p is sublinear. So, from (3), using triangle inequality, we have

$$p(x+y) = \|f\|_Z (\|x+y\|) \le \|f\|_Z (\|x\| + \|y\|)$$

$$= \|f\|_Z \|x\| + \|f\|_Z \|y\| = p(x) + p(y).$$

Hence, is subadditive. Again, we see that $\exists$ a scalar $\alpha \in \mathbb{K}$ X

$p(\alpha x) = \|f\|_Z \|\alpha x\| = \alpha \|f\|_Z \|x\| = \alpha\, p(x)$, proving the absolute homogeneously of p.

Therefore, p is sublinear.

Consequently, by the generalized Hahn - Banach theorem there exists a linear functional $\tilde{f}$ on X which is an extension of f from Z to X, and satisfy

$$|\tilde{f}(x)| \le p(x) = \|f\|_Z \|x\|, \quad \forall x \in X$$

i.e. $|\tilde{f}(x)| \le \|f\|_Z \|x\|; \quad \forall x \in X$ ——— (4)

Taking supremum over all x of norm one in (4), produces

$$\|\tilde{f}\|_X = \sup_{\|x\|=1} |\tilde{f}(x)| \le \|f\|_Z$$

i.e. $\|\tilde{f}\|_X \le \|f\|_Z$ ——— (5)

since under extension, the norm function cannot decrease, it follows, therefore from ③ that

$$\|\tilde{f}\|_X \geq \|f\|_Z \quad \text{———} \quad ⑥$$

from ⑤ and ⑥ we conclude that

$$\|\tilde{f}\|_X = \|f\|_Z \quad \square.$$

Exercise

1. Show that if X is a vector space over the field $\mathbb{K}$, then a subadditive real-valued functional p on X satisfies: $|p(x)-p(y)| \leq p(x-y)$, $\forall x, y \in X$.

2. If $f(x) = f(y)$ for every bounded linear functional f on a normed space X, then show that $x = y$.

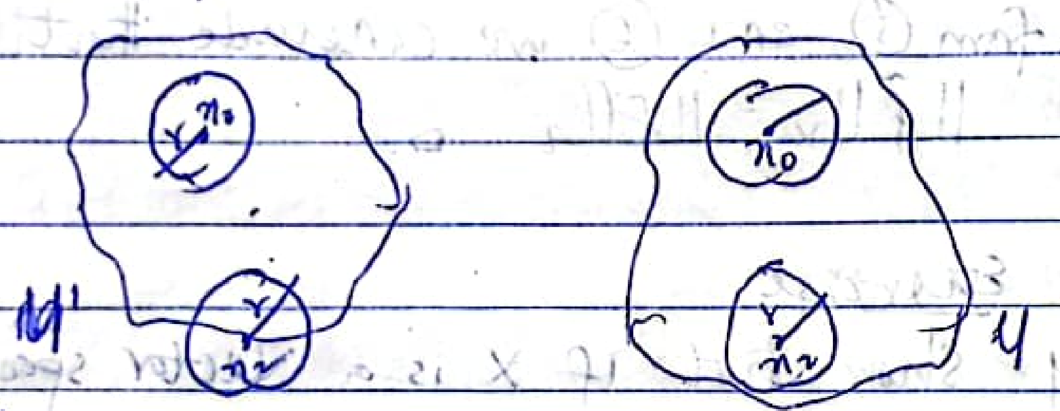
Topic. Principle of Uniform Boundedness.

Definition

Let X be a metric space. A point $x_0 \in M \subset X$ is called an interior point of M if there is an open ball centered at x_0 that lies entirely in M.

Definition

A set M is open if each of its point is an interior point.



Definition

A subset M of a metric space X is said to be:

i. rare (nowhere dense) in X if its closure $\bar{M}$ has no interior point.

ii. Meager (or of the first category) in X if M is the union of countably many sets which are rare in X.

iii. nonmeager (or of the second category) in X if M is not meager in X.

Baire's Category theorem

If a non-empty metric space is complete, then it is nonmeager in itself. Hence, if $X \neq \phi$ is complete, and

$X = \bigcup_{k=1}^{\infty} A_k$, where A_k is closed for each k, then atleast one of A_k contains a nonempty open subset of X.

<u>Uniform Boundedness Theorem / principle</u>

Let (T_n) be a sequence of bounded linear operator $T_n : X \to Y$ from a banach space X into a normed space Y such that $(\|T_n\|)$ is bounded for every $x \in X$, say

$$\|T_n x\| \le \alpha_x, \quad \forall n \in \mathbb{N} \quad \text{--- (1)}$$

where α_x (which varies with x) is a real numbers.
Then the sequence of the norm $\|T_n\|$ is bounded, that is, there exists a real number α s.t.

$$\|T_n\| \le \alpha, \quad \forall n \in \mathbb{N} \quad \text{--- (2)}$$

proof

For every $p \in \mathbb{N}$, let $A_p \subset X$ be the set of all $x \in X$ such that

$$\|T_n x\| \le p, \quad \forall n \in \mathbb{N}.$$

clearly, A_p is a closed set. To see this, notice for every $x \in \bar{A}_p$, there exists a sequence $\{x_i\}_{i \in \mathbb{N}}$ in A_p such that $x_i \xrightarrow{i \to \infty} x$.

Clearly, T_n is bounded and linear, It is therefore continuous. so, T_n is sequentially continuous on X.

Hence, $T_n x_i \xrightarrow[i\to\infty]{} T_n x$. By the continuity of the norm, we have

$$\|T_n x\| = \|\lim_{i\to\infty} T_n x_i\|$$

$$= \lim_{i\to\infty} \|T_n x_i\| \leq p$$

i.e. $\|T_n x\| \leq p, \ \forall n \in \mathbb{N}$.

This shows that $x \in A_p$, and A_p is closed. By ①, each $x \in X$ belongs to some A_p.

Hence

$$X = \bigcup_{p=1}^{\infty} A_p$$

Since X is complete as a metric space, therefore Baire's category theorem implies that each A_p contains an open ball, say

$$B_r(x_0) \subset A_{p_*} \quad \text{——} \quad ③$$

Let x be an arbitrary nonzero element of X, and take $z = x_0 + \eta x$, $\eta = r/2\|x\|$ —— ④

then, $\|z - x_0\| = \|\eta x\| = \left\|\frac{r x}{2\|x\|}\right\| = \frac{r}{2}\frac{\|x\|}{\|x\|} = r/2$.

i.e. $\|z - x_0\| = r/2 \Rightarrow \|z - x_0\| < r$

$\therefore z \in B_r(x_0) \subset A_{p_*}$. Hence, by the definition of A_{p_*}, it follows that $\|T_n z\| \leq p_*$. Similarly

$$\|T_n x_0\| \leq p_*.$$

From ④, $x = \frac{1}{\eta}(z - x_0)$

therefore,

$$\|T_n x\| = \|T_n(\tfrac{1}{\eta}(z - z_0))\| = \tfrac{1}{\eta}\|T_n(z - x_0)\|$$
$$\le \tfrac{1}{\eta}[\|T_n z\| + \|T_n x_0\|] \le \tfrac{1}{\eta}(2P_*) \quad \text{--- (5)}$$

From $\eta = r/2\|x\|$, we have $1/\eta = 2\|x\|/r$. Hence (5) becomes

$$\|T_n x\| \le \frac{2\|x\|}{r}(2P_*) = \frac{4P_*}{r}\|x\| \quad \text{--- (6)}$$

i.e. $\|T_n x\| \le \frac{4}{r}P_x\|x\|$ ——— (6)

Taking supremum over all x of norm one in (6) gives

$$\|T_n\| = \sup_{\|x\|=1} \|T_n x\| \le \frac{4}{r}P_* = \alpha \text{ (say)}$$

i.e. $\|T_n\| \le \alpha$, $\forall n \in \mathbb{N}$, proving the uniform boundedness of T_n. $\square$